TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 5, Pages 1759–1781
S 0002-9947(02)02902-1
Article electronically published on January 10, 2002

POLAR AND COISOTROPIC ACTIONS ON KÄHLER MANIFOLDS

FABIO PODESTÀ AND GUDLAUGUR THORBERGSSON

ABSTRACT. The main result of the paper is that a polar action on a compact irreducible homogeneous Kähler manifold is coisotropic. This is then used to give new examples of polar actions and to classify coisotropic and polar actions on quadrics.

1. INTRODUCTION

The aim of the present paper is to investigate the relationship between polar and coisotropic actions on compact Kähler manifolds.

The action of a compact Lie group G of isometries on a Riemannian manifold (M, g) is called *polar* if there exists a properly embedded submanifold Σ which meets every G-orbit and is orthogonal to the G-orbits in all common points. Such a submanifold Σ is called a *section* (see [PT1],[PT2]) and if it is flat, the action is called *hyperpolar*. It is of course meaningful to relax the definition of polar actions and not require that the section be properly embedded. Our results are correct for weakly polar actions in this sense, making Theorems 1.1 and 1.3 stronger, but Theorem 1.2 weaker.

If (M, g) is a compact Kähler manifold with Kähler form ω and G is a compact subgroup of its full isometry group, then the G-action is called *multiplicity-free* ([GS2]) or *coisotropic* ([HW]) if the principal G-orbits are coisotropic with respect to ω. Notice that the existence of one coisotropic principal G-orbit implies the same property for all principal G-orbits, see [HW].

In this paper we shall consider the special case of M being a simply connected, compact homogeneous Kähler manifold; such manifolds are diffeomorphic to quotient spaces L/K, where L is a compact semisimple Lie group and K is the centralizer in L of some torus. We also recall that a compact homogeneous Kähler manifold cannot be written as a nontrivial Riemannian product of two Riemannian manifolds if and only if its full isometry group is a compact simple Lie group; in this last case the manifold is simply connected (see e.g. [On], p. 238 ff).

In section 2 we will prove the following

Theorem 1.1. *A polar action on an irreducible compact homogeneous Kähler manifold is coisotropic.*

Received by the editors November 8, 2000 and, in revised form, July 31, 2001.
2000 *Mathematics Subject Classification.* Primary 53C55, 57S15.
Key words and phrases. Polar and coisotropic actions, homogeneous Kähler manifolds.
Part of the work on this paper was done during a visit of the second author to the University of Florence and was financially supported by G.N.S.A.G.A. - I.N.d.A.M.

As will be explained in the remark after the proof of Lemma 2.7, a large part of the proof of Theorem 1.1 holds true without requiring irreducibility and homogeneity of the Kähler manifold; there we also explain how one can replace the condition of homogeneity.

It is obvious that Theorem 1.1 is not true without the assumption that M is irreducible. The converse of the theorem is not true. In fact, we will give an example of a coisotropic action on $\mathbb{P}_n(\mathbb{C})$ which is not polar in section 3 and there are further such examples on quadrics in Theorem 1.3. In certain situations, we can prove that coisotropic actions are polar and we will use this to give many new examples of polar actions. We will for example show that every nonoriented compact surface N is the section of a polar action on some four dimensional compact Kähler manifold. In section 3 we will prove the following

Theorem 1.2. *An effective and isometric action of the (real) torus* T^n *on a compact Kähler manifold* M *with positive Euler characteristic and complex dimension* n *is polar.*

The condition on the Euler characteristic is to ensure that the action is Poisson; see the proof in section 3. We also remark that this condition is not restrictive, since a compact, symplectic $2n$-dimensional manifold which is acted on effectively by an n-dimensional torus in a Poisson fashion has positive Euler characteristic (see [De]).

As an application of Theorem 1.1, we will classify polar actions on quadrics in section 4. We do this by first classifying the coisotropic actions and then deciding which of them are polar. An interesting consequence of this classification is that polar actions on quadrics are hyperpolar. This is in contrast to complex projective spaces or more generally to rank one symmetric spaces that admit many polar actions that are not hyperpolar (see [PTh]). Another interesting consequence of the classification is that all coisotropic actions on quadrics are related to transitive actions on spheres. The classification of coisotropic and polar actions on quadrics is given in the following

Theorem 1.3. *Let* G *be a connected compact Lie subgroup of* $\mathrm{SO}(n)$ $(n \geq 5)$ *and let* Q_{n-2} *be the quadric* $\mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-2)$*. Then the action of* G *on* Q_{n-2} *is nontransitive and coisotropic if and only if one of the following occurs.*

(i) *The action of* G *on* $\mathbb{R}^n$ *is irreducible and* G *is conjugate to*

$$\mathrm{SU}(\frac{n}{2}),\ \mathrm{U}(\frac{n}{2}),\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(\frac{n}{4}),\ \mathrm{T}^1\cdot\mathrm{Sp}(n/4),\ \mathrm{Spin}(9).$$

Only the action of the last one of these groups is not polar on Q_{n-2}*.*

(ii) *The action of* G *on* $\mathbb{R}^n$ *has a one dimensional fixed point set and the group* G *as a subgroup of* $\mathrm{SO}(n-1)$ *is conjugate to one of the following:*

$$\begin{aligned}&\mathrm{SO}(n-1),\ \mathrm{G}_2,\ \mathrm{Spin}(7),\\ &\mathrm{SU}(\frac{(n-1)}{2}),\ \mathrm{U}(\frac{(n-1)}{2}),\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(\frac{(n-1)}{4}),\\ &\mathrm{T}^1\cdot\mathrm{Sp}(\frac{(n-1)}{4}),\ \mathrm{Spin}(9).\end{aligned}$$

Only the actions of the first three of these groups are polar on Q_{n-2}*.*

(iii) *The action of* G *on* $\mathbb{R}^n$ *is reducible and* $\mathbb{R}^n$ *is the sum of two irreducible nontrivial* G*-submodules* V_1, V_2*; the group* G *splits as the product* $G_1 \times G_2$

with $G_i \subset \mathrm{SO}(V_i)$, $i = 1, 2$, and each G_i is conjugate in $\mathrm{SO}(V_i)$ to one of the following

$$\mathrm{SO}(p),\ \mathrm{G}_2,\ \mathrm{Spin}(7),\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(p),\ \mathrm{Spin}(9)$$

for suitable $p \in \mathbb{N}$. Only the actions obtained choosing each G_i, $i = 1, 2$, among the first three of these groups are polar on Q_{n-2}.

Moreover if the G-action on Q_{n-2} is polar, then it is hyperpolar.

We now recall some basic definition and results from the paper [HW]. A submanifold W of a symplectic manifold (M, ω) is called *coisotropic* if

$$T_pW^{\perp_\omega} \subseteq T_pW$$

for all p in W. Here $T_pW^{\perp_\omega}$ denotes the subspace of T_pM that is ω-orthogonal to T_pW. If M is a Kähler manifold and ω its Kähler form, then it is easy to see that a submanifold W of M is coisotropic if and only if

$$J(N_pW) \subseteq T_pW$$

for all p in W, where J denotes the complex structure of M, and N_pW the normal space of W in p. One calls a real subspace V in a complex vector space with a Hermitian scalar product *totally real* if V is perpendicular to $J(V)$. It follows from this discussion that a submanifold W in a Kähler manifold is coisotropic if and only if all its normal spaces are totally real.

Let (M, ω) be a symplectic manifold. Then one can associate to every function f in $C^\infty(M)$ its symplectic gradient X_f defined as the unique vector field on M such that

$$df(Y) = \omega(X_f, Y)$$

for every vector field Y on M. If f and g are two functions in C^∞, then one defines their *Poisson bracket* as $\{f, g\} = \omega(X_f, X_g)$. It follows that $C^\infty(M)$ with the Poisson bracket is a Lie algebra.

Now let G act on M preserving the symplectic structure ω. Then the action is called *Poisson* if there is a Lie algebra homomorphism $\lambda : \mathfrak{g} \to C^\infty(M)$ such that $X_{\lambda(\xi)}$ agrees with the infinitesimal action of ξ on M. The *moment map* of a Poisson action is defined as

$$\Phi : M \to \mathfrak{g}^*; \quad \Phi(p)(\xi) = \lambda(\xi)(p).$$

We have $\Phi(gp) = \mathrm{Ad}^*(g)\Phi(p)$, i.e., the moment map is G-equivariant.

Let G be a compact group acting isometrically on a compact Kähler manifold. This action is automatically holomorphic by a theorem of Kostant (see [KN], vol. I, p. 247) and it induces by compactness of M an action of the complexified group $G^{\mathbb{C}}$ on M. We say that M is $G^{\mathbb{C}}$-*almost homogeneous* if $G^{\mathbb{C}}$ has an open orbit in M. If all Borel subgroups of $G^{\mathbb{C}}$ act with an open orbit on M, then the open orbit Ω in M under the action of $G^{\mathbb{C}}$ is called a *spherical homogeneous space* and M is called a *spherical embedding of* Ω. In general, given a complex Lie group U and a closed subgroup $H \subset U$, the pair (U, H) is called a *spherical pair* if every Borel subgroup of U has an open orbit in U/H; all such pairs (U, H) where U is semisimple and H is reductive were classified by Brion ([Br2]) generalizing previous work of Krämer ([Kra]), who assumed U to be simple.

One of the basic results on spherical embeddings is that if the orbit Ω of $G^{\mathbb{C}}$ is a spherical homogeneous space, then $G^{\mathbb{C}}$ and all Borel subgroups of $G^{\mathbb{C}}$ have finitely many orbits in M, see [Br1].

The following theorem is proved in [HW], p. 275.

Theorem 1.4 (Equivalence Theorem of [HW])**.** *Let M be a connected compact Kähler manifold with an isometric action of a connected compact group G that is also Poisson. Then the following conditions are equivalent:*

(i) *The space $C^\infty(M)^G$ of G-invariant functions on M is abelian with respect to the Poisson bracket.*
(ii) *The G-action is coisotropic.*
(iii) *The cohomogeneity of the G-action is equal to the difference between the rank of G and the rank of a regular isotropy subgroup of G.*
(iv) *The moment map $\Phi : M \to \mathfrak{g}^*$ separates orbits.*
(v) *The Kähler manifold M is projective algebraic, $G^{\mathbb{C}}$-almost homogeneous and a spherical embedding of the open $G^{\mathbb{C}}$-orbit.*

We remark here that the conditions (i) to (iii) are equivalent even without the hypothesis of compactness for M (see [HW]).

Under the hypothesis of Theorem 1.4, the following convexity theorem holds (see [GS1] and [Kir]): if T denotes a maximal torus of G, $\mathfrak{t}^*$ is the fixed point set of T in $\mathfrak{g}^*$ and $\mathfrak{t}^*_+$ denotes a fundamental domain for the action of the Weyl group on $\mathfrak{t}^*$, then the image $\Phi(M)$ under the moment mapping Φ intersects $\mathfrak{t}^*_+$ in a convex polytope.

On the other hand, if G is a compact group of isometries of a Riemannian manifold and if the G-action is polar, then we can associate to any section Σ a finite group W, also called Weyl group, which is defined as $W := N_G(\Sigma)/Z_G(\Sigma)$; here $N_G(\Sigma)$ (resp. $Z_G(\Sigma)$) denotes the subgroup of G given by those elements which map Σ onto itself (resp. act trivially on Σ). It is well known that the intersection of a G-orbit with Σ coincides with a W-orbit and we can identify the two orbit spaces M/G and Σ/W; for general properties of the Weyl group we refer to [PT1], [PTh], [Co].

From Theorem 1.1 and Theorem 1.4 we obtain the following

Corollary 1.5. *Let M be an irreducible homogeneous Kähler manifold which is acted on polarly by a compact Lie group of isometries G. Then*

1. *the cohomogeneity of the G-action does not exceed the rank of G; moreover the complexified group $G^{\mathbb{C}}$ and any Borel subgroup of $G^{\mathbb{C}}$ act on M with finitely many orbits;*
2. *if $\Phi : M \to \mathfrak{g}^*$ denotes the moment map, Σ is any section for the G-action and W its Weyl group, then the moment map induces a homeomorphism between Σ/W and the convex polytope $\Phi(M) \cap \mathfrak{t}^*_+$.*

To prove the corollary one only has to observe that the action of G is Poisson since the full group of isometries of M acts in a Poisson fashion: indeed any flag manifold is a coadjoint orbit whose moment map is simply the inclusion.

We end this section pointing out one common property of polar and coisotropic linear actions of a compact Lie group G which will be used in section 4. It is known that a polar linear action of G on a (real) vector space does not contain two distinct irreducible equivalent G-submodules (see [HPTT], Proposition 2.10); if G is realized as a subgroup of $\mathrm{U}(N)$ for some $N \geq 1$ and if the corresponding action on $\mathbb{C}^N$ is coisotropic with respect to the standard symplectic form of $\mathbb{C}^N$, then every irreducible complex G-module appears in $\mathbb{C}^N$ with multiplicity at most one, i.e. every linear coisotropic action of G is *multiplicity free* (see for instance [HW], Theorem 1, p. 275). We also note here that the restriction of a linear coisotropic action to a complex G-submodule is still coisotropic (see the Equivariant Mapping Lemma in [HW], p. 272); as a consequence, a linear coisotropic action has no nontrivial fixed vectors, in contrast with the polar actions.

The following result states that a coisotropic linear action is multiplicity free also from the 'real' point of view.

Proposition 1.6. *Let G be a compact subgroup of $\mathrm{U}(N)$ for some $N \geq 1$. If the corresponding action of G on $\mathbb{C}^N$ is coisotropic with respect to the standard symplectic form on $\mathbb{C}^N$, then the real G-module $\mathbb{C}^N \simeq \mathbb{R}^{2N}$ does not contain two distinct equivalent G-submodules.*

The proof of this proposition will be given in the next section.

2. Proofs of Theorem 1.1 and Proposition 1.6

Let M be a compact, irreducible homogeneous Kähler manifold with Kähler metric g and complex structure J; if we denote by L the identity component of the full isometry group of (M, g), then L preserves the complex structure J (see the introduction) and we can represent M as $M = L/K$, where K is the centralizer in L of a torus. Note again that the irreducibility of (M, g) implies that L is simple.

We consider a compact connected Lie subgroup $G \subset L$ of holomorphic isometries and we shall suppose that the G-action on M is polar with a section Σ.

We saw in the introduction that a submanifold of a Kähler manifold is coisotropic if and only if all of its normal spaces are totally real. The proof of Theorem 1.1 is therefore a direct consequence of the following proposition.

Proposition 2.1. *The section Σ is totally real.*

The proof of this proposition requires some lemmas.

We assume that the cohomogeneity of the action is at least two since the claim in the proposition is trivially true for cohomogeneity one actions. First of all, if Σ is a section, we may define the complex subspace

$$\mathcal{H}_p = T_p\Sigma \cap JT_p\Sigma \tag{2.1}$$

for $p \in \Sigma$. We next prove that $\mathcal{H}$ is a complex subbundle over Σ. It suffices to show that if p and q are two points in Σ and σ a path in Σ joining them, then parallel transport along σ sends $\mathcal{H}_p$ to $\mathcal{H}_q$. This is clear since Σ is totally geodesic and parallel transport in M commutes with J. An important consequence of this argument that we will use later is that $\mathcal{H}$ is a parallel subbundle of $T(\Sigma)$.

We can now define a complex subbundle $\mathcal{H}$ of $T(M)|_{M_{\mathrm{reg}}}$ by setting $\mathcal{H}_q = T_q\Sigma(q) \cap JT_q\Sigma(q)$ for $q \in M_{\mathrm{reg}}$, where $\Sigma(q)$ denotes the unique section through q and M_{reg} denotes the set of regular points of the G-action.

Our aim is to prove that $\mathcal{H}_p = \{0\}$ for all p. In order to do this, we will first show that $\mathcal{H}$ extends to a smooth integrable subbundle of the whole $T(M)$, whose orthogonal complement is also integrable, and then we will use a result on foliations to prove our claim.

Lemma 2.2. *The subbundle $\mathcal{H}$ can be extended to a differentiable subbundle of the tangent bundle $T(M)$.*

Proof. We consider a nonregular point $q \in M$ and fix two sections Σ, Σ' passing through q. If we denote by $\mathcal{H}_q(\Sigma)$ and $\mathcal{H}_q(\Sigma')$ the complex subspaces of $T_q\Sigma$ and $T_q\Sigma'$ respectively, defined as in (2.1), we first must prove that they coincide and then that the extension of the bundle to the whole manifold M is differentiable. $\square$

We will use the fact that the isotropy representation of G_q is polar (see [PT1], [PT2]). We start proving the following

Sublemma 2.3. *The subspace $\mathcal{H}_q(\Sigma)$ is contained in the intersection of all singular hyperplanes of $T_q\Sigma$ with respect to the isotropy representation of G_p.*

Proof. Indeed, if $Y \subset T_q\Sigma$ is a singular hyperplane, then there exists an element $g \in G_q$ which leaves Σ invariant and such that dg_q is a reflection in Y. Since g is holomorphic, it leaves the maximal holomorphic subspace of $T_q\Sigma$ invariant; moreover, since the fixed point set Y has real codimension one, it must contain $\mathcal{H}_q(\Sigma)$. $\square$

Sublemma 2.4. *Let G be a compact Lie group acting linearly and polarly on a vector space V. If Σ and Σ' are two different sections with nontrivial intersection, then the intersection $\Sigma \cap \Sigma'$ contains the intersection of all singular hyperplanes of Σ (and of Σ').*

Proof. We use induction on the dimension of V, the case when $\dim V = 2$ being trivially true. First of all, we may suppose that the fixed point set V^G is equal to $\{0\}$. Indeed, if $V^G \neq \{0\}$, we decompose $V = V^G \oplus W$, with $W = (V^G)^\perp$ and we note that any section splits as the sum of V^G plus a section for the G-action in W; using the induction hypothesis on W, we get our claim. So, we suppose that $V^G = \{0\}$. If we fix a nonzero vector $v \in \Sigma \cap \Sigma'$, we clearly have that v is singular for the G-action and moreover the orbit Gv has positive dimension. Therefore the isotropy G_v acts polarly on the normal space N_v to the orbit Gv and, since $\dim N_v < \dim V$, by the induction hypothesis, we have that $\Sigma \cap \Sigma'$ contains the intersection of all singular hyperplanes in Σ passing through v. A fortiori, it then follows that $\Sigma \cap \Sigma'$ contains the intersection of all singular hyperplanes in Σ. $\square$

We now extend the bundle $\mathcal{H}$ to the singular points. We fix a singular point q and consider two sections Σ, Σ' through q. They correspond to two linear sections, also denoted by Σ, Σ', in the normal spaces N_q of the orbit Gq for the polar action of the isotropy subgroup G_q. The K-cycles of Bott and Samelson ([BS]) can be used to prove that there is a sequence of linear sections $\Sigma_0 = \Sigma, \dots, \Sigma_l = \Sigma'$ such that $\Sigma_i \cap \Sigma_{i+1}$ is nontrivial since we are assuming cohomogeneity greater than one (see the proof of Lemma 1B.3 in [PTh]). From Sublemma 2.4, we see that $\Sigma_i \cap \Sigma_{i+1}$ contains the intersection of all singular hyperplanes of Σ_i and therefore, by Sublemma 2.3, it contains $\mathcal{H}_q(\Sigma_i)$; similarly, it contains $\mathcal{H}_q(\Sigma_{i+1})$. Since $\mathcal{H}_q(\Sigma_i)$ is the maximal complex subspace of Σ_i, it follows that $\mathcal{H}_q(\Sigma_{i+1}) \subseteq \mathcal{H}_q(\Sigma_i)$ and we also have the opposite inclusion, so that $\mathcal{H}_q(\Sigma_i) = \mathcal{H}_q(\Sigma_{i+1})$. Therefore $\mathcal{H}_q(\Sigma) = \mathcal{H}_q(\Sigma')$.

We are left with proving that the extension is differentiable.

We fix a singular point $q \in M$ and a basis $\{v_1, \dots, v_l\}$ of $\mathcal{H}_q$; we will show that there exist smooth vector fields $v_1^*, \dots, v_l^*$ on some suitable neighborhood U of q, which extend $v_1, \dots, v_l$ respectively and span $\mathcal{H}_y$ for all $y \in U$.

We denote by N_q the normal space $T_q(G \cdot q)^\perp$ and choose a real positive number $r > 0$ so that $\exp_q|_{B_r} : B_r \to M$ is a diffeomorphism onto its image, where

$B_r := \{v \in N_q;\ ||v|| < r\}$. We set $\mathcal{N}_r := \exp_q(B_r)$ and note that $\mathcal{N}_r = \bigcup \mathcal{N}_r \cap \Sigma$, where the union is taken over all sections Σ passing through q. If $y \in \mathcal{N}_r$, then $y = \exp_q(Y)$ for some $Y \in B_r$ and for $i = 1, \ldots, l$ we define $\hat{v}_i(y)$ to be the parallel transport of v_i along the geodesic $\exp_q(tY)$, $t \in [0, 1]$; note that if Σ is any section through q with $Y \in T_q\Sigma$, then $y \in \Sigma$ and $\hat{v}_i(y) \in \mathcal{H}_y \subseteq T_y\Sigma \subset T_y\mathcal{N}_r$, since $\mathcal{H}$ is parallel along the sections.

Now let $\mathfrak{g}_q$ be the Lie algebra of the isotropy subgroup G_q and fix a complement $\mathfrak{m}_q$ of $\mathfrak{g}_q$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}_q + \mathfrak{m}_q$; it is clear that, if r is sufficiently small, there exists a neighborhood V of $O \in \mathfrak{m}_q$ so that the map

$$V \times \mathcal{N}_r \to M$$
$$(X, y) \mapsto \exp^G(X) \cdot y$$

is a diffeomorphism onto an open neighborhood U of q in M. If $x \in U$, say $x = \exp^G(X) \cdot y$ with $X \in V$ and $y \in \mathcal{N}_r$, we define $v_i^*(x) = \exp^G(X)_*(\hat{v}_i(y)) \in \mathcal{H}_x$ for all $i = 1, \ldots, l$ and this proves our claim.

We now prove the following

Sublemma 2.5. *The distribution given by $\mathcal{H}$ is integrable and totally geodesic; the distribution given by $\mathcal{H}^\perp$ is integrable.*

Proof. It is enough to prove our claims on the regular set M_{reg} which is dense in M.

The claims about $\mathcal{H}$ follow from the fact that $\mathcal{H}$ is a parallel subbundle of $T(\Sigma)$ for all sections Σ.

We now put $k = \dim \Sigma - \dim \mathcal{H}$ and m the dimension of a regular G-orbit. Then $\dim \mathcal{H}^\perp = k + m$. We pick a regular point q and consider the section Σ through q; we then fix a set $\{Z_1, \ldots, Z_k\}$ of smooth independent sections of $\mathcal{H}^\perp$ which are tangent to Σ over a suitable neighborhood $V \cap \Sigma$, where V is an open neighborhood of q in M_{reg}. We may extend the sections $\{Z_1, \ldots, Z_k\}$ to vector fields on a neighborhood of q by using the G-action, since the slice representation of G_p is trivial for every $p \in V$. On some suitable open neighborhood V' of q in V, we can choose Killing vector fields $\{X_1, \ldots, X_m\}$, induced by the G-action, which span the tangent space to every orbit Gp for $p \in V'$; the vector fields $\{Z_1, \ldots, Z_k, X_1, \ldots, X_m\}$ are therefore linearly independent and span $\mathcal{H}_p^\perp$ for all $p \in V'$. We now prove that $\mathcal{H}^\perp$ is integrable. First of all we note that, by the definition of Z_i the Lie derivative $\mathcal{L}_{X_j} Z_i = 0$ and therefore $[Z_i, X_j] = 0$ for all $i = 1, \ldots, k,\ j = 1, \ldots, m$; moreover, $[X_i, X_j]$ is tangent to the G-orbits and is therefore a section of $\mathcal{H}^\perp$ for all $i, j = 1, \ldots, m$. It is left to prove that $[Z_i, Z_j]$ are sections of $\mathcal{H}^\perp$ for all $i, j = 1, \ldots, k$.

Given any section $X \in \Gamma(\mathcal{H})$ on V', we have

$$g([Z_i, Z_j], X) = g(Z_i, \nabla_{Z_j} X) - g(Z_j, \nabla_{Z_i} X),$$

where ∇ denotes the Levi-Civita connection of the metric g. If $x \in V'$ and $\Sigma(x)$ denotes the (unique) section through x, then Z_i, Z_j and X are tangent to $\Sigma(x)$ and $\nabla_{Z_i} X, \nabla_{Z_j} X$ are sections of $\mathcal{H}$, since $\mathcal{H}$ is a parallel subbundle of $T(\Sigma(x))$; therefore $g([Z_i, Z_j], X) = 0$ and our claim follows. □

Lemma 2.6. *We have $\mathcal{H}_p = \{0\}$ for all $p \in M$.*

Proof. In [BH] the following result is proved: given a totally geodesic foliation $\mathcal{F}$ on a compact Riemannian manifold (M, g) such that the distribution given by $T\mathcal{F}^\perp$ is integrable, then the universal covering manifold $\tilde{M}$ splits topologically as the product of two manifolds that are the universal coverings of leaves of the two distributions $\mathcal{F}$ and $\mathcal{F}^\perp$. In our situation, the manifold M is simply connected; moreover, it is known (see [On], p. 279 and [Sh]) that the cohomology ring $H^*(M,\mathbb{R})$ is indecomposable, i.e. can not be written as a tensor product $A\otimes B$, where A, B are differential graded algebras of dimension bigger than two. This implies that the leaves of the distribution $\mathcal{H}$ are zero dimensional. □

Proof of Proposition 2.1. Lemma 2.6 shows that for every $p\in\Sigma$ we have $JT_p\Sigma\cap T_p\Sigma=\{0\}$; we now have to prove that Σ is a totally real submanifold, that is $JT_p\Sigma\perp T_p\Sigma$ for every $p\in\Sigma$.

Since the manifold M is compact, the complexified Lie group $G^{\mathbb{C}}$ acts on M. Moreover, we can consider an Iwasawa decomposition $\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}+\mathfrak{s}$, where $\mathfrak{s}$ is a solvable Lie subalgebra. We recall that the manifold $M=L/K$ can be $L^{\mathbb{C}}$-equivariantly embedded into some complex projective space. If S denotes the Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{s}$, then by the Borel Fixed Point Theorem, S has a fixed point in M and therefore there exists a point $q\in M$ such that the $G^{\mathbb{C}}$-orbit $G^{\mathbb{C}}q$ coincides with the G-orbit Gq, which is therefore complex. We denote by N_q the normal space of the orbit Gq at q; then N_q is a complex subspace of T_qM and the isotropy G_q acts on it polarly with a section Σ.

First of all, we note that the fixed point set $N_q^{G_q}$ is equal to $\{0\}$, since otherwise it would be a complex subspace contained in a section, contradicting Lemma 2.6. Therefore the G_q-module N_q splits as a sum of irreducible, nontrivial complex submodules; these are also irreducible over the reals since otherwise N_q would contain two equivalent real submodules, which is not possible by [Da], see also [HPTT], p. 168. Our claim will now follow from the following lemma. □

Lemma 2.7. *Let K be a compact Lie subgroup of $\mathrm{O}(V)$, where V is a real vector space. If the action of K on V is irreducible, polar and leaves some complex structure J on V invariant, then the sections are totally real subspaces with respect to J.*

Proof. We know from Dadok's Theorem that the K-action is orbit equivalent to the isotropy representation of a symmetric space U/H and that K can be considered to be a subgroup of H. Furthermore the representation of K on V is the restriction of the isotropy representation of U/H. If K itself is the isotropy subgroup of a symmetric space U/K, then U/K is Hermitian symmetric and the sections are totally real: indeed if $\mathfrak{u}=\mathfrak{k}+\mathfrak{m}$ is the corresponding Cartan decomposition, we identify V with $\mathfrak{m}$ and the complex structure J can be represented as the adjoint action of some element $z\in\mathfrak{k}$; therefore, if Σ is any section and $v,w\in\Sigma$, then $\langle Jv,w\rangle=\langle[z,v],w\rangle=0$, since w belongs to Σ which is orthogonal to the tangent space of the orbit $\mathrm{Ad}(K)$-orbit through v.

If, on the other hand, there exists a symmetric pair (U,H) such that K is a proper subgroup of H and H,K have the same orbits in V, where $\mathfrak{u}=\mathfrak{h}+V$ is the corresponding Cartan decomposition, then the pairs (K,H) have been completely classified in [EH] (see also [GT]). We note here that our claim is obviously true when the cohomogeneity is one. The list of such pairs (K,H) with cohomogeneity of the K-action at least two is given in the following table, where we also indicate the group U, such that (U,H) is a symmetric pair.

K	H	U
$\mathrm{SO}(2)\times \mathrm{G}_2$	$\mathrm{SO}(2)\times\mathrm{SO}(7)$	$\mathrm{SO}(9)$
$\mathrm{SO}(2)\times\mathrm{Spin}(7)$	$\mathrm{SO}(2)\times\mathrm{SO}(8)$	$\mathrm{SO}(10)$
$\mathrm{SO}(3)\times\mathrm{Spin}(7)$	$\mathrm{SO}(3)\times\mathrm{SO}(8)$	$\mathrm{SO}(11)$
$\mathrm{SU}(p)\times\mathrm{SU}(q),\ p\neq q$	$S(\mathrm{U}(p)\times\mathrm{U}(q))$	$\mathrm{SU}(p+q)$
$\mathrm{SU}(n),\ n$ odd	$\mathrm{U}(n)$	$\mathrm{SO}(2n)$
$\mathrm{SO}(10)$	$\mathrm{T}^1\cdot\mathrm{SO}(10)$	E_6

Since the corresponding symmetric spaces U/H are Hermitian symmetric with the only exception of $H = \mathrm{SO}(3)\times\mathrm{SO}(8)$ and $K = \mathrm{SO}(3)\times\mathrm{Spin}(7)$, it will be enough to prove that the representation space $V = \mathbb{R}^{24}$ of the group K (here the representation is the tensor product of the standard representation of SO(3) on $\mathbb{R}^3$ and the spin representation of Spin(7) on $\mathbb{R}^8$) does not carry any invariant complex structure. If V carries such a complex structure, then K would have an irreducible representation in $\mathbb{C}^{12}$, which would be a tensor product of two irreducible representations of SO(3) and Spin(7) respectively; this contradicts the fact that the least dimensional irreducible complex representation of Spin(7) is seven dimensional. □

Remark. We point out here that the homogeneity of the Kähler manifold M was used only in the proof of Lemma 2.6 and at the beginning of the proof of Proposition 2.1 where we use the Borel Fixed Point Theorem.

In fact neither homogeneity nor irreducibility was used to show the existence of the bundle $\mathcal{H}$, its integrability and the integrability of its orthogonal bundle. In particular looking at the proof of Lemma 2.6, we see that Theorem 1.1 holds true for every compact, projective algebraic Kähler manifold whose universal covering space is not diffeomorphic to a product of two Kähler manifolds of lower dimension.

Proof of Proposition 1.6. We denote by J the complex structure and by $\langle,\rangle$ the scalar product on the real G-module $\mathbb{R}^{2N}$. We split $\mathbb{C}^N = \sum_{i=1}^k W_i$ as the sum of mutually inequivalent complex irreducible G-submodules; we recall that each $(W_i)_\mathbb{R}$, i.e. the real G-module obtained from W_i by restricting the scalars, is either irreducible or splits as the sum of two equivalent submodules which are interchanged by J. Suppose that the G-module $\mathbb{R}^{2N}$ contains two equivalent irreducible G-submodules; then we can suppose that there exist two equivalent G-submodules E_1, E_2 such that either E_1, E_2 are both complex or $E_1 + E_2$ is complex and J interchanges E_1, E_2. Since the restriction of the G-action to a complex submodule is still coisotropic we can restrict ourselves to the complex submodule $W := E_1 + E_2$. We identify both E_1, E_2 with the irreducible G-module V endowed with the invariant scalar product g and W with the orthogonal sum $V \oplus V$ so that $\langle,\rangle$ restricts to g on the first factor and with $c^2 g$ on the second one for $c \in \mathbb{R}\setminus\{0\}$. If E_1 and E_2 are not complex submodules, then $Hom(V,V)^G = \mathbb{R}$ (see e.g. [BtD], p. 99) and it is easy to see that J is given up to the sign by $J(v,w) = (-cw, \frac{1}{c}v)$ for $v, w \in V$; if both E_1, E_2 are complex submodules, then V has a complex structure $\tilde{J}$ and $J(v,w) = (\tilde{J}v, -\tilde{J}w)$, since $E_1 \simeq \overline{E_2}$ as complex modules.

We now construct two functions $f_1, f_2 \in C^\infty(W)^G$ which do not commute with respect to the Poisson bracket, contradicting Theorem 1.4. We define $f_1(v,w) = g(v,v)$ and $f_2(v,w) = g(v,w)$ for $v, w \in V$. Then $X_{f_1}|_{(v,w)}$ is given by $(2\tilde{J}v, 0)$ if V is complex and by $(0, \frac{2}{c}v)$ if $JE_1 = E_2$. It is then easy to check that $\{f_1, f_2\} = -df_2(X_{f_1})$ does not vanish identically, proving our claim. □

3. New examples of polar actions

In this section we first prove Theorem 1.2 and then give new examples of polar actions. We also show that there is a coisotropic action which is not polar.

Proof of Theorem 1.2. We will denote by J and ω the complex structure and the Kähler form of M respectively and denote the torus T^n by G. The commutativity of G implies that the action has only one principal isotropy group which is contained in all other isotropy groups. It follows that the principal isotropy group is trivial since the action is effective. Hence the principal orbits are n-dimensional. Since the Euler characteristic of M is positive, the G-action has a fixed point and it is therefore Poisson by a result of Frankel [Fr]; this together with the commutativity of G implies that $\omega(X,Y) = 0$ for all $X, Y \in T_p(Gp)$ and therefore $JT_p(Gp) = N_p(Gp)$ for every regular point p, where $N_p(Gp)$ denotes the normal space of Gp at p. We denote by Φ the corresponding moment map, which separates G-orbits by Theorem 1.4.

Let $P = \Phi(M)$ be the moment polytope of the T^n-space M. Delzant constructs in [De] a 'canonical' smooth projective variety M_P that is G-equivariantly symplectomorphic to M. The symplectic manifold M_P is naturally equipped with a Kähler metric g_P, a complex structure J_P and a symplectic form ω_P; moreover there exists an antiholomorphic involution τ_P on M_P such that $\tau_P g = g^{-1}\tau_P$ for all $g \in G$ and $\Phi\tau_P = \Phi$. We denote by Q_P the fixed point set of τ_P. The compact, properly embedded submanifold Q_P is not empty: indeed, if $z \in M_P^G$, the fixed point set of G in M_P, then $\tau_P z \in M_P^G$ and $\Phi\tau_P z = \Phi z$. Since Φ separates orbits, we have $\Phi^{-1}\Phi w = w$ for all $w \in M_P^G$. Hence $\tau_P z = z$.

It follows from [Du] that $\Phi(Q_P) = \Phi(M_P)$ since Q_P is not empty. Hence Q_P contains regular points; if $q \in Q_P$ is regular, then we have the orthogonal splitting $T_q M_P = T_q(Gq) \oplus T_q Q_P$ (with respect to g_P) and $J_P(T_q Q_P) = T_q(Gq)$.

Abreu shows in [Ab] that the complex manifolds (M,J) and (M_P, J_P) are G-equivariantly biholomorphic. If ϕ is such a biholomorphism, then $\tau := \phi^{-1}\tau_P\phi$ is an antiholomorphic involution of M (which is not necessarily isometric). If Q denotes the fixed point set of τ, namely $Q := \phi^{-1}(Q_P)$, then for every regular point $q \in Q$ we have $T_q Q = JT_q(Gq) = N_q(Gq)$. This shows that a connected component Σ of Q is an integral submanifold for the normal distribution to the regular orbits. It is a well known fact that Σ meets every G-orbit; moreover Σ intersects the regular and hence all G-orbits orthogonally. □

We will now give examples of torus actions, in which we can explicitly describe the sections.

Example 1. We consider the complex projective space $\mathbb{P}_n(\mathbb{C})$ together with the polar action of the n-dimensional torus T^n, which is simply the maximal torus in the group $\mathrm{SU}(n+1)$; the regular orbits are n-dimensional tori and any section is a totally geodesic real projective space $\mathbb{P}_n(\mathbb{R})$.

Our aim is to produce new examples of polar actions on nonhomogeneous Kähler manifolds, which are not hyperpolar. In order to do this, we choose a T^n-fixed point $p \in \mathbb{P}_n(\mathbb{C})$ and we denote by M_1 the complex manifold which is obtained blowing up $\mathbb{P}_n(\mathbb{C})$ in p; note that M_1 is a projective variety and that M_1 is not homogeneous with respect to the full group of holomorphic automorphisms (hence with respect to the full group of isometries of any Kähler metric compatible with the given complex structure), see [Bl]. The T^n-action lifts to a holomorphic action on M_1; moreover, since M_1 is Kähler and T^n is compact, we can find a T^n-invariant Kähler metric g on M_1 by averaging; we remark that the metric g has *a priori* no relation with the Fubini-Study metric on $\mathbb{P}_n(\mathbb{C})$. It follows from Theorem 1.2 that the action of T^n on M_1 is polar. In the following we will determine the section of the action.

We denote by $\pi : M_1 \to \mathbb{P}_n(\mathbb{C})$ the holomorphic projection. We fix a section Σ in $\mathbb{P}_n(\mathbb{C})$ and denote by Σ_1 the closed submanifold of M_1 which coincides with Σ outside $\{p\}$ and has the property that $\Sigma_1 \cap \pi^{-1}(p)$ is $\mathbb{P}_{\mathbb{C}}(T_p\Sigma)$, where we have identified $\pi^{-1}(p)$ with $\mathbb{P}_{\mathbb{C}}(T_p\mathbb{P}_n(\mathbb{C}))$. It is clear that Σ_1 meets every T^n-orbit. We are now going to show that Σ_1 is a section for the T^n-action with respect to the Kähler metric g and we will prove this in the dense open set of regular points.

First of all we remark that the T^n-action on M_1 is a Poisson action, since the manifold M_1 is simply connected (see e.g. [GS3]); moreover the T^n-action is coisotropic by Theorem 1.4, (v). We denote by J the complex structure on M_1 and fix a regular point $y \in M_1 \setminus \pi^{-1}(p)$. If we denote by N_y the normal space to the orbit $\mathrm{T}^n \cdot y$, then $JN_y = T_y(\mathrm{T}^n \cdot y)$ since the action is coisotropic. On the other hand, $J(T_y\Sigma_1)$ also coincides with $T_y(\mathrm{T}^n \cdot y)$, so that $N_y = T_y(\Sigma_1)$ and therefore Σ_1 is a section. We remark that the section Σ_1, as a differentiable manifold, is the connected sum $\mathbb{P}_n(\mathbb{R})\#\mathbb{P}_n(\mathbb{R})$.

Since the T^n-action on M_1 has fixed points in $\pi^{-1}(p)$, we can iterate this construction blowing up M_1 in one of these fixed points. In this way, we get a sequence of compact Kähler manifolds M_k on which the group T^n acts polarly with the sections being the connected sum of $k+1$ copies of $\mathbb{P}_n(\mathbb{R})$. If $n \geq 3$, then $\pi_1(\Sigma_1)$ is a free group $\mathbb{Z}_2 * \mathbb{Z}_2 * \ldots * \mathbb{Z}_2$ ($k+1$ times). Therefore it can not carry a flat metric by the Bieberbach Theorem (see [Wo]).

Remark. We point out that the lifted action of T^n on M_k is polar with respect to *any* invariant Kähler metric. We also remark that this construction yields polar actions on four dimensional Kähler manifolds with any nonorientable compact surface as a section.

We would now like to give examples of polar actions of nonabelian Lie groups using the method of blow-ups.

Example 2. We first describe the general principle which will allow us to obtain new examples. We consider a compact Lie group G acting polarly on a homogeneous Kähler manifold M with at least one fixed point p; we denote by K the isotropy subgroup of a regular point $q \in M$. We recall the following well-known facts on moment maps (see [GS3] and also [HW]). Denote by $\Phi : M \to \mathfrak{g}^*$ the moment map with respect to the given Poisson-Kähler action of G and recall that Φ maps G-orbits onto coadjoint orbits in $\mathfrak{g}^*$; moreover, for any regular point $y \in M$, the isotropy subgroup $H_y := G_{\Phi(y)}$ contains G_y and, at the level of Lie algebras, we may decompose $\mathfrak{h}_y$ (with respect to a fixed bi-invariant scalar product on $\mathfrak{g}$) as

$$\mathfrak{h}_y = \mathfrak{g}_y + \mathfrak{a}_y,$$

where $\mathfrak{a}_y = \mathfrak{g}_y^\perp \cap \mathfrak{h}_y$ is abelian and centralizes $\mathfrak{g}_y$. If we denote by N_y the normal space to the regular orbit $G \cdot y$ and by J the complex structure on M, then

$$J(N_y) = T_y(H_y \cdot y) \subset T_y(G \cdot y),$$

since the G-action on M is coisotropic by Theorem 1.1. It follows that the dimension of $\mathfrak{a}_y$ is equal to the cohomogeneity of the action of G. The tangent space to the orbit $G\cdot y$ can be identified with the complement $\mathfrak{g}_y^\perp$ and we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_y + \mathfrak{a}_y + \mathfrak{m}_y,$$

where $\mathfrak{m}_y$ corresponds to the maximal complex subspace of the tangent space $T_y(G\cdot y)$. In the spirit of S. Lie and F. Engel ([LE], p. 501 ff), we call the G-action $\mathbb{C}$-*asystatic* if for a regular orbit G/K there is no nontrivial complex subspace of the tangent space $T_{[eK]}(G/K)$ which is left fixed by the isotropy representation of K. (For a modern treatment of asystatic actions, see [AA] and [PT1].) It follows that if a polar action as above is also $\mathbb{C}$-asystatic, then the space of all tangent vectors to a regular orbit which are fixed under the isotropy representation has dimension equal to the cohomogeneity of the G-action.

We now assume that the G-action is $\mathbb{C}$-asystatic and we consider the blow-up M_1 of M in the fixed point q. If Σ denotes a section for the G-action on M, then the submanifold Σ_1 of M_1 constructed exactly as in Example 1 is a closed submanifold which meets every G-orbit in M_1. If we now endow M_1 with *any* G-invariant Kähler metric g, we show that Σ_1 is a section, namely that it meets every G-orbit orthogonally. Indeed, the lifted action of G on M_1 is Poisson, since M_1 is simply connected; moreover the G-action is coisotropic because M_1 is a projective algebraic variety, $G^{\mathbb{C}}$ acts on M, hence on M_1, with an open orbit and the same is true for a Borel subgroup of $G^{\mathbb{C}}$ (see Theorem 1.4). We fix a regular point $y \in M_1$ and we denote by $\tilde{N}_y$ the normal space to $T_y(G\cdot y)$ with respect to g, so that $J\tilde{N}_y \subseteq T_y(G\cdot y)$; since the action of G on M is also coisotropic, we have that $J(T_y\Sigma_1) \subseteq T_y(G\cdot y)$ and we claim that $J(\tilde{N}_y) = J(T_y\Sigma_1)$. Both subspaces $J(\tilde{N}_y)$ and $J(T_y\Sigma_1)$ are fixed under the isotropy representation of G_y and have the same dimension, namely the cohomogeneity of G and therefore, by the assumption that the action is $\mathbb{C}$-asystatic, they coincide.

As an application, we consider the quadric $M := Q_n = \mathrm{SO}(n+2)/\mathrm{SO}(2)\times\mathrm{SO}(n)$. The isotropy subgroup $G = \mathrm{SO}(2)\times\mathrm{SO}(n)$ acts polarly on Q_n with sections being two dimensional flat tori. It is easy to check that a regular isotropy subgroup K is given by $K = \mathbb{Z}_2 \times \mathrm{SO}(n-2)$ and therefore the isotropy representation splits as $2\mathbb{R} + 2\mathbb{R}^{n-2}$, where $\mathbb{R}$ is a trivial submodule, while K acts on $\mathbb{R}^{n-2}$ in the standard way; therefore the G-action is $\mathbb{C}$-asystatic. The blown up manifold M_1 is acted on polarly by G with sections being the connected sum $T^2\#\mathbb{P}_2(\mathbb{R})$. Note that M_1 is not homogeneous, as already pointed out in Example 1.

Example 3. We will now give examples of coisotropic action on complex projective spaces that are not polar. Let G denote the compact Lie group $\mathrm{U}(2)\times\mathrm{Sp}(n)$, $n \geq 2$, acting on $\mathbb{C}^{4n} = \mathbb{C}^2\otimes\mathbb{C}^{2n}$ as a tensor product of the standard representations of $\mathrm{U}(2)$ and $\mathrm{Sp}(n)$ respectively. This representation is not polar. (It has cohomogeneity three and is shown to have taut orbits in [GT], but these properties will not be important here.) We consider the complexification $G^{\mathbb{C}} = \mathrm{GL}(2,\mathbb{C})\times\mathrm{Sp}(n,\mathbb{C})$ of G and the corresponding representation on $\mathbb{C}^{4n}$ which is multiplicity free by the classification of such irreducible representations of reductive noncompact complex Lie groups in [Ka]. This means that $G^{\mathbb{C}}$ acts on $\mathbb{C}^{4n}$ with an open orbit as well as every Borel subgroup of $G^{\mathbb{C}}$. The induced action of $G^{\mathbb{C}}$ on $\mathbb{P}_{4n-1}(\mathbb{C})$ clearly also has the property that every Borel subgroup of $G^{\mathbb{C}}$ acts on $\mathbb{P}_{4n-1}(\mathbb{C})$ with an open orbit. This implies by Theorem 1.4 that the action of G on $\mathbb{P}_{4n-1}(\mathbb{C})$ is coisotropic. This action is not polar since otherwise the action of $\mathrm{T}^1\times G$ on $\mathbb{C}^{4n}$, where T^1 acts as multiplication by scalars, would have to be polar by [PTh]; since the center of

G already acts on $\mathbb{C}^{4n}$ as multiplication by scalars by Schur Lemma, $\mathrm{T}^1 \times G$ and G have the same orbits and therefore the G-action on $\mathbb{P}_{4n-1}(\mathbb{C})$ is not polar.

It is not difficult to classify coisotropic actions on complex projective spaces. If G is a compact subgroup of $\mathrm{U}(n+1)$, then the action of G on $\mathbb{P}_n(\mathbb{C})$ is coisotropic if and only if the Borel subgroups of $\mathbb{C}^* \times G^{\mathbb{C}}$ have dense orbits in $\mathbb{C}^{n+1}$ which in turn is equivalent to the action of $\mathbb{C}^* \times G^{\mathbb{C}}$ being multiplicity free on $\mathbb{C}^{n+1}$ (see e.g. [Kr], p. 199). One can now use the classification of multiplicity free representations of reductive groups that can be found in [Ka] for the irreducible case and in [BR] and [Le] for the reducible case.

4. Polar and coisotropic actions on quadrics

The aim of this section is to prove Theorem 1.3, or more precisely to classify coisotropic and polar actions on the Hermitian symmetric space $Q_{n-2} = \mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-2)$ ($n \geq 5$), which can be represented as the $(n-2)$-dimensional complex quadric in $\mathbb{P}_{n-1}(\mathbb{C})$, i.e., the set of all totally isotropic lines in $\mathbb{C}^n$ with respect to the invariant symmetric bilinear form $\langle , \rangle$ given by $\langle z, w \rangle = \sum_{i=1}^n z_i w_i$, or equivalently as the Grassmannian of oriented two-planes in $\mathbb{R}^n$.

Hyperpolar actions of compact Lie groups on irreducible compact symmetric spaces were classified by Kollross ([Ko]). We state his result in the special case of complex quadrics in the following theorem.

Theorem 4.1 (Kollross). *Let $G \subset \mathrm{SO}(n)$ be a compact connected Lie subgroup acting on the symmetric space $Q_{n-2} = \mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-2)$ in a hyperpolar fashion. Then there exists a compact connected Lie subgroup G' of $\mathrm{SO}(n)$ containing G such that the G'-action on Q_{n-2} has the same orbits as the G-action and G' is conjugate to one of the following subgroups:*

(1) $G' = \mathrm{SO}(p) \times \mathrm{SO}(q) \subset \mathrm{SO}(p+q)$ *with* $p, q \geq 1$*;*
(2) $G' = \mathrm{U}(n) \subset \mathrm{SO}(2n)$*;*
(3) $G' = \mathrm{Sp}(1) \cdot \mathrm{Sp}(n) \subset \mathrm{SO}(4n)$*;*
(4) $G' = \mathrm{Spin}(9) \subset \mathrm{SO}(16)$.

Remark. The actions listed in Theorem 4.1 all have cohomogeneity one except case (1) when $p, q > 1$ which has cohomogeneity two. Recall that a compact Lie subgroup of $\mathrm{SO}(n)$ acts transitively on Q_{n-2} if and only if $G = \mathrm{SO}(n)$ with the two exceptions $G = \mathrm{G}_2$ ($n = 7$) and $G = \mathrm{Spin}(7)$ ($n = 8$) as can be seen from [On], Theorem 2, p. 226.

We now consider a compact connected Lie subgroup G of $\mathrm{SO}(n)$ acting coisotropically on Q_{n-2}. The complexification $G^{\mathbb{C}}$ acts on Q_{n-2} with an open orbit Ω and there exists a Borel subgroup of $G^{\mathbb{C}}$ which has an open orbit in Ω.

In order to prove Theorem 1.3, we will make extensive use of the results by Kimelfeld ([Ki]), who classified all reductive subgroups $G^{\mathbb{C}}$ of $\mathrm{SO}(n, \mathbb{C})$ having an open orbit on the quadric Q_{n-2}.

We will distinguish between the two cases when $G \subset \mathrm{SO}(n)$ acts irreducibly on $\mathbb{R}^n$ and when G acts reducibly on $\mathbb{R}^n$.

First case: the G-action on $\mathbb{R}^n$ is irreducible. In this case the complexified action of $\mathfrak{g}^{\mathbb{C}}$ on $\mathbb{C}^n$ can be irreducible or reducible (as the sum of two submodules $W \oplus W^*$, where $W \simeq \mathbb{C}^{n/2}$). These two subcases appear as separate cases in Kimelfeld's

Table I.

n	$\mathfrak{g}^{\mathbb{C}}$	Represen.	$\mathfrak{h}$
1	$\mathfrak{so}(s), s \geq 3$	Simplest	$(\mathfrak{so}(s-1) \oplus \mathfrak{t}_1) + U_{s-2}$
2	$C_s \oplus A_1, s \geq 2$	$R(\phi_1) \times R(\phi_1)$	$(C_{s-2} \oplus A_1 \oplus \mathfrak{t}_1) + U_{4s-5}$
3	G_2	$R(\phi_1)$	A_2
4	B_3	$R(\phi_3)$	G_2
5	B_4	$R(\phi_4)$	B_3
6	A_2	$R(\phi_1 + \phi_2)$	$\mathfrak{t}_2$
7	C_2	$R(2\phi_1)$	$\mathfrak{t}_2$
8	G_2	$R(\phi_2)$	$\mathfrak{t}_2$
9	A_1	$R(4\phi_1)$	0
10	C_3	$R(\phi_2)$	$A_1 \oplus A_1 \oplus A_1$
11	F_4	$R(\phi_1)$	D_4
12	$C_s \oplus C_2, s \geq 2$	$R(\phi_1) \times R(\phi_1)$	$C_{s-2} \oplus A_1 \oplus A_1$
13	$A_1 \oplus A_1$	$R(\phi_1) \times R(3\phi_1)$	0

paper (see Tables I and II in [Ki], pp. 535-536) and we too will reproduce them separately in Tables I and II in this paper.

As for notation, we will use the standard symbols A_s, B_s, C_s, D_s $(s \geq 1)$ and G_2, F_4 to denote complex simple Lie algebras; by $R(\phi)$ we will denote the irreducible representation of a simple Lie algebra with highest weight ϕ and ϕ_i will indicate the highest weight of the ith fundamental representation of a simple complex Lie algebra (the standard representation of $\mathfrak{so}(n)$ on $\mathbb{C}^n$ will be call simplest); by $\mathfrak{t}_n$ we will indicate the Lie algebra of the n-dimensional complex torus and ϵ will denote the faithful one-dimensional representation of $\mathfrak{t}_1$.

In Table I we list all complex reductive Lie algebras $\mathfrak{g}^{\mathbb{C}}$ with the corresponding complex irreducible representation such that the induced action on the quadric Q has an open orbit Ω; moreover we will also indicate a generic stability subalgebra $\mathfrak{h}$, with p-dimensional nilradical U_p, so that $\Omega = G^{\mathbb{C}}/H$, where H is an appropriate subgroup of $G^{\mathbb{C}}$ having $\mathfrak{h}$ as Lie algebra.

We start by noting that the cases 1, 3 and 4 correspond to transitive actions of the compact real form G, while the cases 2 and 5 are known as cohomogeneity one actions (see Kollross' theorem).

In the cases 6 to 8, the group G acts on $\mathfrak{g}$ via adjoint representation; the cohomogeneity of the G-action on the corresponding quadric Q is easily seen to be at least four, hence bigger than the rank of G. This means that the G-action on Q is not coisotropic by Theorem 1.4.

We are now going to exclude all other cases from the above table; in order to do this, we use the fact that a Borel subgroup of $G^{\mathbb{C}}$ has an open orbit in $G^{\mathbb{C}}/H$, see Theorem 1.4. This immediately excludes cases 9 and 13.

The following result (see [Ki], p. 572 ff) is well known:

Lemma 4.2. *Given a semisimple complex algebraic group U and a Borel subgroup B, a reductive subgroup H of U has an open orbit in U/B if and only if the space of H-fixed vectors in every irreducible U-module is at most one-dimensional.*

Note that, if a Borel subgroup B of $G^{\mathbb{C}}$ has an open orbit in $G^{\mathbb{C}}/H$, then H has an open orbit in $G^{\mathbb{C}}/B$.

Table II.

n	$\mathfrak{g}^{\mathbb{C}}$	Represen.	$\mathfrak{h}$
1	$A_s, s \geq 2$	$R(\phi_1)$	$(A_{s-2} \oplus \mathfrak{t}_1) + U_{2s-1}$
2	$A_s \oplus \mathfrak{t}_1, s \geq 2$	$R(\phi_1) \times \epsilon$	$(A_{s-2} \oplus \mathfrak{t}_2) + U_{2s-1}$
3	$A_s \oplus A_1, s \geq 2$	$R(\phi_1) \times R(\phi_1)$	$A_{s-2} \oplus \mathfrak{t}_1$
4	$A_s \oplus A_1 \oplus \mathfrak{t}_1, s \geq 2$	$R(\phi_1) \times R(\phi_1) \times \epsilon$	$A_{s-2} \oplus \mathfrak{t}_2$
5	$\mathfrak{so}(s) \oplus \mathfrak{t}_1$	Simplest $\times\, \epsilon$	$\mathfrak{so}(s-2)$
6	$A_1 \oplus \mathfrak{t}_1$	$R(2\phi_1) \times \epsilon$	0
7	$B_3 \oplus \mathfrak{t}_1$	$R(\phi_3) \times \epsilon$	A_2
8	$G_2 \oplus \mathfrak{t}_1$	$R(\phi_1) \times \epsilon$	A_1
9	$C_s, s \geq 3$	$R(\phi_1)$	$(C_{s-2} \oplus \mathfrak{t}_1) + U_{4s-5}$
10	$C_s \oplus \mathfrak{t}_1, s \geq 3$	$R(\phi_1) \times \epsilon$	$(C_{s-2} \oplus \mathfrak{t}_2) + U_{4s-5}$
11	A_4	$R(\phi_2)$	$A_1 \oplus A_1$
12	$A_4 \oplus \mathfrak{t}_1$	$R(\phi_2) \times \epsilon$	$A_1 \oplus A_1 \oplus \mathfrak{t}_1$
13	D_5	$R(\phi_5)$	A_3
14	$D_5 \oplus \mathfrak{t}_1$	$R(\phi_5) \times \epsilon$	$A_3 \oplus \mathfrak{t}_1$

In the remaining cases 10 to 12, we note that these are complexifications of isotropy representations of rank two symmetric spaces and that $\mathfrak{h}$ is exactly the complexification of a regular stability subalgebra for these representations. Since H stabilizes a two-dimensional subspace of the corresponding irreducible representation of $G^{\mathbb{C}}$, the G-action on the quadric is not coisotropic by Theorem 1.4, (v) and Lemma 4.2.

We now come to the subcase that the complexified action of $\mathfrak{g}^{\mathbb{C}}$ on $\mathbb{C}^n$ is reducible (as the sum of two submodules $W \oplus W^*$, where $W \simeq \mathbb{C}^{n/2}$). In the following Table II, we reproduce all such cases having an open orbit on the corresponding quadric, only indicating the irreducible representation W and the generic stability subalgebra $\mathfrak{h}$ for the $G^{\mathbb{C}}$-action on the corresponding quadric; we will indicate by U_p the p-dimensional nilradical of the generic stability subalgebra $\mathfrak{h}$.

We start by noting that case 2 is of cohomogeneity one (see Theorem 4.1) and the same holds for case 1: indeed, the action of $\mathrm{U}(s)$ on $\mathrm{SO}(2s)/\mathrm{SO}(2) \times \mathrm{SO}(2s-2)$ has a complex singular orbit which is $\mathbb{P}_{s-1}(\mathbb{C}) = \mathrm{U}(s)/\mathrm{U}(1) \times \mathrm{U}(s-1)$; this orbit also is an orbit of the semisimple part $\mathrm{SU}(s)$ and the stabilizer $S(\mathrm{U}(1) \times \mathrm{U}(s-1))$ still acts with cohomogeneity one on the normal space.

Cases 6 to 8 and 11 to 14 can be ruled out simply because the dimension of a Borel subgroup of $G^{\mathbb{C}}$ is strictly less than the dimension of the open orbit $G^{\mathbb{C}}/H$.

In cases 3 and 4 we consider the action of the compact group G acting on the quadric $Q := Q_{4s+2}$ and we will prove that in case 4 this G-action is not coisotropic; by Theorem 1.4 this implies that the G-action in case 3 is also not coisotropic. We may take G as $S(\mathrm{U}(2) \times \mathrm{U}(s+1))$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^{s+1} \simeq \mathbb{R}^{4s+4}$; we then fix two unit vectors $e_1 \in \mathbb{C}^2$, $v \in \mathbb{C}^{s+1}$ and we take the oriented two plane π in $\mathbb{R}^{4s+4}$ given by the complex line $\mathbb{C} \cdot e_1 \otimes v$. It is easy to see that G_π is isomorphic to $S(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(s))$, so that the orbit G/G_π is complex and biholomorphic to $S^2 \times \mathbb{P}_s(\mathbb{C})$. Now, if V denotes the subspace $(\mathbb{C} \cdot v)^\perp \subset \mathbb{C}^{s+1}$ and e_2 is a unit vector generating $(\mathbb{C} \cdot e_1)^\perp \subset \mathbb{C}^2$, then we have

$$T_\pi Q = Hom(\pi, \pi^\perp) = \pi^* \otimes_{\mathbb{R}} [(e_1 \otimes V)_{\mathbb{R}} + (e_2 \otimes V)_{\mathbb{R}} + (\mathbb{C} \cdot e_2 \otimes v)_{\mathbb{R}}],$$

which contains two pairs of equivalent $2s$-dimensional G_π-submodules. Since the isotropy representation of G_π on $T_{eG_\pi}G/G_\pi$ contains only one $2s$-dimensional submodule, it then follows that the slice representation of G_π at π contains at least two distinct equivalent submodules. If the G-action on Q is coisotropic, then the slice representation of G_π is also coisotropic (see [HW], p. 274), contradicting Proposition 1.6.

Case 5 can be ruled out because the pair $(\mathfrak{so}(s) + \mathfrak{t}_1, \mathfrak{so}(s-2))$ is not a spherical pair; indeed, if it were spherical, the pair $(\mathfrak{so}(s), \mathfrak{so}(s-2))$ would be spherical too, contradicting Lemma 4.2.

In case 9, we show that the action of $G = \mathrm{Sp}(s)$ is not coisotropic. We consider the G-action on $\mathbb{H}^s \simeq \mathbb{C}^{2s} \simeq \mathbb{R}^{4s}$ and the corresponding action on the quadric $Q := Q_{4s-2}$. We fix an oriented two plane π in $\mathbb{R}^{4s}$, which is invariant under the complex structure of $\mathbb{C}^{2s}$. Then $G_\pi = \mathrm{T}^1 \times \mathrm{Sp}(s-1)$ and the orbit $G \cdot \pi$ is diffeomorphic to a complex projective space $\mathbb{P}_{2s-1}(\mathbb{C})$. The tangent space $T_\pi Q$ can be identified with $Hom(\pi, \pi^\perp)$ and we have the orthogonal splitting $\pi^\perp = \hat{\pi} + W$, where $\pi + \hat{\pi}$ is the quaternionic span of π, while $W \simeq \mathbb{R}^{4(s-1)}$ is the standard representation space of $\mathrm{Sp}(s-1)$.

If we denote by ρ the representation of the center T^1 of G_π on π, then T^1 acts on $\hat{\pi}$ by means of ρ^{-1}; the tangent space $T_\pi Q$ splits as a G_π-module as $2\mathbb{R} + \mathbb{R}^2 + 2W$, where $\mathbb{R}$ denotes the trivial module, $\mathbb{R}^2$ is acted on by T^1 by means of ρ^2, while W is the standard representation space of G_π. Since the isotropy representation of G_π on the orbit $G \cdot \pi$ is $\mathbb{R}^2 + W$, we see that the normal space N_π is isomorphic to $2\mathbb{R} + W$ as a G_π-module; it follows that a regular isotropy subgroup K of the G-action on Q is isomorphic to $\mathrm{T}^1 \times \mathrm{Sp}(s-2)$ of corank one in G, while the G-cohomogeneity on Q is three. This proves that the G-action on Q is not coisotropic.

In case 10 the group $G = \mathrm{T}^1 \times \mathrm{Sp}(s)$ acts on $\mathbb{C}^{2s} \simeq \mathbb{R}^{4s}$ in a standard way; for notational reasons we will denote by Z the one-dimensional center of G. We will show that the G-action on $Q := Q_{4s-2}$ is coisotropic but not polar. As in the previous case, we fix an oriented two plane $\pi \subset \mathbb{R}^{4s}$ which is Z-invariant; the isotropy subgroup G_π is $Z \times \mathrm{T}^1 \times \mathrm{Sp}(s-1)$ and the orbit $\mathcal{O} := G \cdot \pi$ is a complex projective space $\mathbb{P}_{2s-1}(\mathbb{C})$. If we compute the slice representation along the same lines as in case 9, we see that the normal space N_π splits as a G_π-module as $\mathbb{R}^2 + \mathbb{R}^{4(s-1)}$, where G_π acts nontrivially on each factor (this action can be shown to be actually polar). A regular isotropy subgroup K is isomorphic to $\mathrm{T}^1 \times \mathrm{Sp}(s-2)$ of corank two in G and the cohomogeneity is two, hence the G-action on Q is coisotropic.

We now show that this action is not polar. It is not difficult to see that there exists a totally geodesic submanifold $F \subset Q$ with $T_\pi F = N_\pi$ such that F is isometric to the complex projective space $\mathbb{P}_{2s-1}(\mathbb{C})$ after an appropriate rescaling of the metrics. Moreover we observe that the group G lies in $\mathrm{U}(2s)$, which acts on Q with cohomogeneity one; the two singular $\mathrm{U}(2s)$-orbits are $\mathcal{O}$ and $\mathcal{O}^* := \mathrm{U}(2s) \cdot \pi^* = G \cdot \pi^*$, where π^* denotes the two plane π with reversed orientation. The orbit $\mathcal{O}^*$ consists actually of all points of maximal distance d from $\mathcal{O}$ and d is the diameter of F. Now suppose that the G-action is polar. If Σ is a section passing through π, then $\Sigma \subset F$ and we know from [PTh] that Σ is a real projective plane. The submanifold Σ contains a circle S of points at distance d from π, hence S is contained in $\mathcal{O}^*$; this means that Σ intersects the G-orbit in a circle, hence not orthogonally; a contradiction.

So far we have proved the following

Proposition 4.3. *Let G be a compact connected Lie subgroup of* $\mathrm{SO}(n)$*, where* $n \geq 5$. *Suppose the action of G on $\mathbb{R}^n$ is irreducible. Then the action of G on $Q := Q_{n-2}$ is nontransitive and polar if and only if it is hyperpolar and* $G = \mathrm{U}(n/2), \mathrm{SU}(n/2), G = \mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$ *or* $G = \mathrm{Spin}(9) \subset \mathrm{SO}(16)$.

The action of G on Q is nontransitive and coisotropic if and only if the action is polar or $G = \mathrm{T}^1 \cdot \mathrm{Sp}(n/4)$

Second case: the G-action on $\mathbb{R}^n$ is reducible. We will now suppose that the linear G-action on $\mathbb{R}^n$ is reducible. Here the proof will be more direct than in the first case. We will first deal with the particular case when G has a fixed nonzero vector e in $\mathbb{R}^n$, so that we can write $\mathbb{R}^n = \mathbb{R} \cdot e + \mathbb{R}^{n-1}$ (note that we are not supposing that $\mathbb{R}^{n-1}$ is G-irreducible). We prove the following

Proposition 4.4. *Let G be a compact connected Lie group acting on $V := \mathbb{R}^n$ with nonzero fixed point set. Then*

(i) *the action of G on $Q := Q_{n-2}$ is coisotropic if and only if the fixed point set V^G is one dimensional and G acts transitively on the unit sphere of $(V^G)^\perp \simeq \mathbb{R}^{n-1}$ with the exception of $G = \mathrm{Sp}(k)$ acting in a standard way on $\mathbb{R}^{4k}, n = 4k+1$.*

(ii) *the action of G on $Q := Q_{n-2}$ is polar if and only if the fixed point set V^G is one dimensional and $G = \mathrm{SO}(n-1), G = \mathrm{G}_2$ $(n = 8)$ or $G = \mathrm{Spin}(7)$ $(n = 9)$; in particular the action of G is polar if and only if it is hyperpolar.*

Proof. We suppose that the G-action on Q_{n-2} is coisotropic. We first note that the fixed point set V^G is one dimensional. Indeed, if $\dim V^G \geq 2$, we select an oriented two plane $\pi \subseteq V^G$ and note that the isotropy representation of $G_\pi = G$ must be coisotropic (see [HW], p. 274, Remark (3)); this means that G acts on $Hom(\pi, \pi^\perp) \cong_G 2\pi^\perp$ coisotropically, contradicting Proposition 1.6. □

Next, we choose a nonzero vector $e \in V^G$ and we split $V = \mathbb{R} \cdot e + W$ as an orthogonal sum; we aim at proving that G acts transitively on the unit sphere of W. We further split W into the sum of G-irreducible nontrivial submodules $W = \sum_{i=1}^k V_i$, $\dim V_i \geq 2, i = 1, \dots, k$. The restriction of the $\mathfrak{g}$-action onto V_i defines homomorphisms $\rho_i : \mathfrak{g} \to \mathfrak{so}(V_i)$ for $i = 1, \dots, k$ and we put $\mathfrak{g}_i := \rho_i(\mathfrak{g})$, $i = 1, \dots, k$. The action of G restricted to V_i induces a coisotropic action on the corresponding quadric and, if $\dim V_i \geq 5$, Proposition 4.3 and an inspection of the list of compact groups acting transitively on quadrics (see the remark after Theorem 4.1) show that $\mathfrak{g}_i$ is one of the following Lie algebras

$$(*) \qquad \mathfrak{so}(p), \mathfrak{G}_2, \mathfrak{spin}(7), \mathfrak{spin}(9), \mathfrak{su}(p), \mathfrak{u}(p), \mathbb{R} + \mathfrak{sp}(p), \mathfrak{sp}(1) + \mathfrak{sp}(p)$$

for some integer p. On the other hand, if $2 \leq \dim V_i \leq 4$, the list in (*) contains all possible subalgebras of $\mathfrak{so}(V_i)$ which act on V_i irreducibly, so that (*) gives the list of all possible candidates for $\mathfrak{g}_i$, $i = 1, \dots, k$.

We now show that $k = 1$. We suppose that $k \geq 2$; we restrict ourselves to $\mathbb{R} \cdot e + V_1 + V_2$ and we denote by Q the corresponding quadric. First we prove the following

Lemma 4.5. *We have* $\mathfrak{g}_i \neq \mathfrak{su}(p), \mathfrak{u}(p), \mathbb{R} + \mathfrak{sp}(p)$.

Proof. We suppose that $\mathfrak{g}_1 \in \{\mathfrak{su}(p), \mathfrak{u}(p), \mathbb{R} + \mathfrak{sp}(p)\}$. We fix an oriented two plane π in V_2 so that the orbit $G \cdot \pi$ is a complex orbit and let $\mathfrak{g}_\pi$ denote the stability subalgebra. If we identify $T_\pi Q$ with $Hom(\pi, \pi^\perp)$, we see that the slice representation at π contains a submodule isomorphic to $Hom(\pi, V_1)$. Let J_0 and J_1 denote $\mathfrak{g}_\pi$-invariant complex structures in π and V_1 respectively and denote by W_k, $k = 1, 2$, the $\mathfrak{g}_\pi$-submodules of $Hom(\pi, V_1)$ consisting of elements f such that $f(J_0 v) = (-1)^k J_1 f(v)$ for all $v \in V_1$. It is easy to see that W_1 and W_2 are equivalent $\mathfrak{g}_\pi$-modules. This implies that the G-action cannot be coisotropic by Proposition 1.6. □

From the previous lemma, we have that $\mathfrak{g}$ has no center. Now we pick nonzero vectors $v_i \in V_i$, $i = 1, 2$, and consider the oriented two planes $\pi_i = e \wedge v_i$; if Z_i denotes $\pi_i^\perp$, then $T_{\pi_i} Q$ can be identified with $e \wedge Z_i + v_i \wedge Z_i$ for $i = 1, 2$. We recall then that the complex structure J on $T_\pi Q$ is given (up to sign) by

$$J(e \wedge w_1 + v \wedge w_2) = e \wedge w_2 - v \wedge w_1,$$

for $w_1, w_2 \in Z_i$, $i = 1, 2$; since $T_{\pi_i} G \cdot \pi_i = e \wedge \mathfrak{g} \cdot v_i$, we have that the orbits $G \cdot \pi_i$ are totally real. This means that the moment map Φ is constant along each orbit $G \cdot \pi_i$ and, since $\mathfrak{g}$ has no center, we have that $\Phi(G \cdot \pi_i) = \{0\}$. But this contradicts the fact that Φ separates orbits (see Theorem 1.4, (iv)). So $k = 1$.

We now use the fact that G induces a coisotropic, possibly transitive action on the subquadric $Q(\mathbb{R}^{n-1}) = Q_{n-3}$ and that $\mathfrak{g}$ is one of the Lie algebras appearing in (*). Then G can be either

(*a*) $G = \mathrm{SO}(n-1)$, G_2, or Spin(7) which all give rise to a cohomogeneity one action on Q (those of G_2 and Spin(7) being orbit equivalent to the one of $\mathrm{SO}(n-1)$ for an appropriate n), or

(*b*) $G = \mathrm{SU}(\frac{n-1}{2})$, $\mathrm{U}(\frac{n-1}{2})$, $\mathrm{Sp}(1) \cdot \mathrm{Sp}(\frac{(n-1)}{4})$, $\mathrm{T}^1 \cdot \mathrm{Sp}(\frac{(n-1)}{4})$ or Spin(9).

It is easy to check that all groups listed in (*b*) yield coisotropic actions; actually their cohomogeneity on the quadric is given by the cohomogeneity of G_v acting on $\mathfrak{g} \cdot v$, where $v \in \mathbb{R}^{n-1}$ denotes a G-regular vector in $\mathbb{R}^{n-1}$.

We will now show that the G-action on the quadric is not polar in all the cases enumerated in (*b*). Now let G be one of the groups listed in (*b*); we already know that the induced action of $\mathrm{T}^1 \cdot \mathrm{Sp}(\frac{(n-1)}{4})$ on the subquadric Q_{n-3} is not polar if $n > 5$, while, for $n = 5$, G coincides with U(2), so that this case can be ruled out. Now, if $v \in \mathbb{R}^{n-1}$ denotes a G-regular vector in $\mathbb{R}^{n-1}$ and $\pi = e \wedge v$, then it is easy to see that

$$T_\pi Q_{n-2} = T_\pi(G \cdot \pi) + v \wedge \mathfrak{g} \cdot v$$

and that G_v acts on $\mathfrak{g} \cdot v$ polarly and with cohomogeneity two. We choose two orthogonal vectors w_1, w_2 of unit length in $\mathfrak{g} \cdot v$ which span a G_v-section in $\mathfrak{g} \cdot v$; if the G-action on Q is polar, then there exists a section Σ passing through π with $T_\pi \Sigma = Span(v \wedge w_1, v \wedge w_2)$. The unique totally geodesic submanifold of Q with such properties is given by $\Sigma = \{v \wedge (ae + bw_1 + cw_2);\ a^2 + b^2 + c^2 = 1\}$. If we now consider the two plane $\pi_1 = v \wedge w_1$, then $T_{\pi_1} \Sigma$ is generated by $v \wedge e, v \wedge w_2$, while $T_{\pi_1} G \cdot \pi_1$ is given by $\mathfrak{g} v \wedge w_1 + v \wedge \mathfrak{g} w_1$. Since the scalar product $\langle v \wedge \mathfrak{g} w_1, v \wedge w_2 \rangle = \langle \mathfrak{g} w_1, w_2 \rangle$ is not zero, we see that Σ cannot be a section and this contradiction proves our last claim.

We now suppose that G acts on $\mathbb{R}^n$ reducibly without fixed vectors and that it acts on the quadric Q_{n-2} coisotropically. We split $\mathbb{R}^n = V_1 \oplus V_2$ into two nontrivial invariant subspaces with $\dim V_i \geq 2$ and consider first the case when both V_i are G-irreducible. The restriction of the $\mathfrak{g}$-action onto V_i defines homomorphisms $\rho_i : \mathfrak{g} \to \mathfrak{so}(V_i)$ for $i = 1, 2$. We put $\mathfrak{g}_i := \rho_i(\mathfrak{g})$, $i = 1, 2$, so that $\mathfrak{g} \subset \mathfrak{g}_1 + \mathfrak{g}_2$. As in the proof of Proposition 4.4, we see that each $\mathfrak{g}_i$ is one of the Lie algebras listed in (*).

We now prove the following

Lemma 4.6. *If $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}_1$, then $\mathfrak{g}_2 \subset \mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}_1$, the kernel of ρ_1, given by $\mathfrak{g} \cap \mathfrak{g}_2$, is not trivial. Therefore we can write $\mathfrak{g} = \hat{\mathfrak{g}}_1 + (\mathfrak{g} \cap \mathfrak{g}_2)$, the sum of two ideals of $\mathfrak{g}$, where $\hat{\mathfrak{g}}_1$ is isomorphic to $\mathfrak{g}_1$. Now $\rho_2(\mathfrak{g} \cap \mathfrak{g}_2)$ is an ideal of $\mathfrak{g}_2$; if $\mathfrak{g}_2$ is simple, then $\rho_2(\mathfrak{g} \cap \mathfrak{g}_2) = \mathfrak{g} \cap \mathfrak{g}_2 = \mathfrak{g}_2$, hence $\mathfrak{g}_2 \subseteq \mathfrak{g}$. If $\mathfrak{g}_2$ is not simple, namely $\mathfrak{g}_2 \simeq \mathfrak{sp}(1) + \mathfrak{sp}(p)$ and $\mathfrak{g}_2 \not\subset \mathfrak{g}$, then $\rho_2(\hat{\mathfrak{g}}_1)$ is a nontrivial ideal of $\mathfrak{g}_2$. We note that $\rho_2(\hat{\mathfrak{g}}_1)$ cannot coincide with $\mathfrak{g}_2$. Indeed, if so, we have $\mathfrak{g}_1 \simeq \mathfrak{sp}(1) + \mathfrak{sp}(p) \simeq \mathfrak{g}_2$ and $\rho_2|_{\hat{\mathfrak{g}}_1}$ is an isomorphism; since $\ker \rho_2 = \mathfrak{g} \cap \mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g} \cap \mathfrak{g}_1 \cap \hat{\mathfrak{g}}_1 = \{0\}$, we have $\ker \rho_2 = \{0\}$ and $\dim \mathfrak{g} = \dim \mathfrak{g}_2 = \dim \mathfrak{g}_1$, forcing $\mathfrak{g} \simeq \mathfrak{g}_1$, a contradiction.

Therefore we have the following possibilities:

(a) $\rho_2(\hat{\mathfrak{g}}_1) = \mathfrak{sp}(1)$. Then $\mathfrak{g} \cap \mathfrak{g}_2 = \mathfrak{sp}(p)$. Now $\hat{\mathfrak{g}}_1 \simeq \mathfrak{g}_1$ contains an ideal isomorphic to $\mathfrak{sp}(1)$, hence we have the following subcases:

(a1) $\mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p)$, where $\rho_1(\mathfrak{g}) = \mathfrak{sp}(1)$ acting on $\mathbb{R}^3$ by Lemma 4.5, $\rho_2(\mathfrak{g}) = \mathfrak{sp}(1) + \mathfrak{sp}(p)$;

(a2) $\mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p) + \mathfrak{sp}(q)$, where $\mathfrak{g}_1 = \mathfrak{sp}(1) + \mathfrak{sp}(q)$ and $\mathfrak{g}_2 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$.

(b) $\rho_2(\hat{\mathfrak{g}}_1) = \mathfrak{sp}(p)$. Then we have the following subcases:

(b1) $\mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p)$, where $\mathfrak{g}_1 = \mathfrak{sp}(p)$, $p = 1, 2$;

(b2) $\mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p) + \mathfrak{sp}(1)$ and $\mathfrak{g}_1 \simeq \mathfrak{g}_2$.

We now show that the above cases do not yield coisotropic actions. Indeed, for (a1), if we fix a two plane $\pi \subset \mathbb{R}^3 = V_1$, then $\rho_2(\mathfrak{g}_\pi) \subset \mathbb{R} + \mathfrak{sp}(p)$ leaves a complex structure invariant and the same arguments as in Lemma 4.5 can be applied. Cases (a2) and (b2) can be dealt with similarly. As for (b1) a simple computation shows that the cohomogeneity c of G is at least 4 if $p = 1$ and 9 if $p = 2$, contradicting the fact that c does not exceed the rank of G. □

Therefore, if $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}_i$ for $i = 1, 2$, then $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$.

Lemma 4.7. *If $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, then*

(i) *the action of G on Q_{n-2} is coisotropic if and only if*

$$\mathfrak{g}_i \in \{\mathfrak{so}(V_i), \mathfrak{G}_2, \mathfrak{spin}(7), \mathfrak{spin}(9), \mathfrak{sp}(1) + \mathfrak{sp}(p)\}$$

for $i = 1, 2$;

(ii) *the action of G on Q_{n-2} is polar if and only if*

$$\mathfrak{g}_i \in \{\mathfrak{so}(V_i), \mathfrak{G}_2, \mathfrak{spin}(7)\}$$

for $i = 1, 2$.

Proof. (i) We use the list (*) together with Lemma 4.5.

Now if $\mathfrak{g}_i \in \{\mathfrak{so}(p), \mathfrak{G}_2, \mathfrak{spin}(7)\}$ for $i = 1, 2$, then it can be easily checked that G has the same orbits in the quadric $Q = Q_{n-2}$ as $\mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$, which acts on Q hyperpolarly, hence coisotropically.

Now we suppose that, say, $\mathfrak{g}_1 = \mathfrak{spin}(9)$. We choose an oriented two plane $\pi \subset V_2$ such that the orbit $G\cdot\pi$ is complex; then the stability subalgebra $\mathfrak{g}_\pi = \mathfrak{spin}(9)+(\mathfrak{g}_2)_\pi$ and $\mathrm{rank}(\mathfrak{g}_2)_\pi = \mathrm{rank}(\mathfrak{g}_2)$ since $G \cdot \pi$ is a flag manifold.

We first claim that the $\mathfrak{g}_\pi$-action on π is not trivial.

Indeed, when $\mathfrak{g}_2$ is $\mathfrak{so}(V_2), \mathfrak{G}_2$ or $\mathfrak{spin}(7)$, then our claim follows from the fact the groups $\mathrm{SO}(V_2), \mathrm{G}_2$ and $\mathrm{Spin}(7)$ act transitively on the Stiefel manifold of two frames in V_2 (see e.g. [On], pp. 90, 93). When $\mathfrak{g}_2 = \mathfrak{spin}(9)$, then the action of $\mathrm{Spin}(9)$ on Q_{14} has cohomogeneity one and we claim that there exists only one complex singular orbit in Q_{14} given by $G\cdot\pi = \mathrm{Spin}(9)/\mathrm{U}(4)$, where $\mathrm{U}(4)$ acts nontrivially on π. Indeed, we see the spin representation of $\mathrm{Spin}(9)$ as the isotropy representation of the Cayley plane $\mathbb{P}_2(\mathbb{O}) = \mathrm{F}_4/\mathrm{Spin}(9)$ at the origin o and, if we choose a two plane π inside the tangent space $W \cong \mathbb{R}^8$ at o of a projective line in $\mathbb{P}_2(\mathbb{O})$ through o, then the stabilizer $\mathrm{Spin}(9)_\pi$ leaves W invariant, since two different projective lines cannot be tangent along a two plane; this means that $\mathrm{Spin}(9)_\pi$ is contained in $\mathrm{Spin}(8)$ and therefore it covers $\mathrm{T}^1 \times \mathrm{SO}(6) \subset \mathrm{SO}(W)$. Hence $\mathrm{Spin}(9)_\pi = \mathrm{U}(4)$ and it acts nontrivially on π. Moreover, since $\chi(Q_{14}) = 16$ is the sum of the Euler characteristics of the singular orbits in Q_{14}, and since $\chi(\mathrm{Spin}(9)/\mathrm{U}(4)) = 16$, we see that there is exactly one complex singular orbit $G \cdot \pi = \mathrm{Spin}(9)/\mathrm{U}(4)$. When $\mathfrak{g}_2 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$, then the action of $\mathrm{Sp}(1) \cdot \mathrm{Sp}(p)$ on Q_{4p-2} has cohomogeneity one; if π is a complex line, viewed as a real two plane, inside a quaternionic line in $\mathbb{H}^p$, then it is easy to see that $(\mathfrak{g}_2)_\pi = \mathbb{R}^2 + \mathfrak{sp}(p-1)$ acts nontrivially on π. The same argument as in the previous case shows that $\mathrm{Sp}(1) \cdot \mathrm{Sp}(p)$ has only one complex singular orbit in Q_{4p-2}.

So we may write $(\mathfrak{g}_2)_\pi = \mathbb{R} + \mathfrak{h}_2$, where $\mathfrak{h}_2$ denotes the kernel of the $(\mathfrak{g}_2)_\pi$-action on π. The slice representation of $(\mathfrak{g}_2)_\pi$ on N_π splits as $(\mathbb{R}^2)^* \otimes V_1 + N'$, where N' is acted on by $\mathbb{R} + \mathfrak{h}_2$; note that $N' = \{0\}$ if and only if $\mathfrak{g}_2 \in \{\mathfrak{so}(p), \mathfrak{G}_2, \mathfrak{spin}(7)\}$. Now $(\mathbb{R}^2)^* \otimes V_1$ is acted on by $\mathbb{R} + \mathfrak{spin}(9)$, while $\mathfrak{h}_2$ acts trivially on it. The representation of $\mathbb{R} + \mathfrak{spin}(9)$ on $(\mathbb{R}^2)^* \otimes V_1$ is coisotropic, see [Ka] or notice that a regular subalgebra is given by $\mathfrak{su}(3)$ of corank three and the representation has cohomogeneity three (see [Ya], p. 324). If $\mathfrak{g}_2 \in \{\mathfrak{so}(p), \mathfrak{G}_2, \mathfrak{spin}(7)\}$, then the G-action on the quadric Q is coisotropic; indeed, a regular isotropy subalgebra of $\mathfrak{g}$ is $\mathfrak{su}(3) + \mathfrak{h}_2$ of corank three in $\mathfrak{g}_\pi$, hence of corank three in $\mathfrak{g}$, while $N' = \{0\}$ implies that the cohomogeneity of the G-action on Q is the cohomogeneity of $\mathbb{R} + \mathfrak{spin}(9)$ acting on $(\mathbb{R}^2)^* \otimes V_1$.

If $\mathfrak{g}_2 = \mathfrak{spin}(9)$, then $(\mathfrak{g}_2)_\pi = \mathbb{R} + \mathfrak{su}(4)$, $N' = \mathbb{R}^8$ and $\mathfrak{su}(4)$ acts on it by cohomogeneity one; hence a regular isotropy subalgebra of $\mathfrak{g}$ is $\mathfrak{su}(3) + \mathfrak{su}(3)$ of corank four, while the cohomogeneity of the G-action is also four.

If $\mathfrak{g}_2 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$ $(p \geq 2)$, then $(\mathfrak{g}_2)_\pi = \mathbb{R}^2 + \mathfrak{sp}(p-1)$ and $N' = \mathbb{R}^{4p-4}$. So $(\mathfrak{g}_\pi) = \mathfrak{su}(3) + \mathbb{R} + \mathfrak{sp}(p-2)$ of corank four in $\mathfrak{g}$ and the cohomogeneity of the G-action on Q is also four.

Next we suppose that $\mathfrak{g}_1 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$. If we argue in the same way as in the case $\mathfrak{g}_2 = \mathfrak{spin}(9)$, we see that we only need to prove that the representation of $\mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(p)$ on $(\mathbb{R}^2)^* \otimes \mathbb{R}^{4p}$ is coisotropic (see also [Ka]). This representation has cohomogeneity three (see [Ya], p. 318); a regular isotropy subalgebra $\mathfrak{k}$ contains a copy of $\mathfrak{sp}(p-2)$ as an ideal, so that we can write $\mathfrak{k} = \mathfrak{k}' + \mathfrak{sp}(p-2)$ and a simple dimension count shows that $\mathfrak{k}' = \mathbb{R}$. So the corank of $\mathfrak{k}$ in $\mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(p)$ is also three and the representation is coisotropic.

(ii) Using the same arguments as in case (i), we only need to show that the representation of $\mathbb{R} + \mathfrak{spin}(9)$ on $\mathbb{R}^2 \otimes \mathbb{R}^{16}$ and of $\mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(p)$ on $\mathbb{R}^2 \otimes \mathbb{R}^{4p}$

are not polar. Actually these representations are neither isotropy representations of irreducible symmetric spaces nor appear in the list of irreducible polar representations which are not isotropy representations of symmetric spaces (see the table from [EH] in the proof of Lemma 2.7). □

Next we suppose that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_1$.

If $\mathfrak{g}_1$ is simple, then ρ_1, ρ_2 are isomorphisms and $\mathfrak{g}_1 \simeq \mathfrak{g}_2$. In particular, $\mathfrak{g}_1 \in \{\mathfrak{so}(V_1), \mathfrak{G}_2, \mathfrak{spin}(7), \mathfrak{spin}(9)\}$. Now, if ρ_1 is equivalent to ρ_2, we select a two plane $\pi \subset V_1$ so that the orbit $G \cdot \pi$ is complex and observe that the slice representation of $\mathfrak{g}_\pi$ contains $Hom(\pi, V_2)$ as a complex submodule; if we denote by π the same oriented two plane in $V_2 \simeq V_1$, then $Hom(\pi, \pi)$ contains a trivial $\mathfrak{g}_\pi$-submodule. This means that the slice representation contains a trivial complex submodule, preventing the G-action from being coisotropic. If on the other hand ρ_1 is not equivalent to ρ_2, then $\mathfrak{g} = \mathfrak{spin}(7)$ or $\mathfrak{g} = \mathfrak{spin}(9)$; in this case a simple computation shows that the dimension of a Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$ is strictly less than the complex dimension of the quadric and the action is therefore not coisotropic.

If $\mathfrak{g}_1$ is not simple, i.e. $\mathfrak{g}_1 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$, then $\mathfrak{g}_2$ can be $\mathfrak{sp}(1) \simeq \mathfrak{so}(3)$, $\mathfrak{sp}(2) \simeq \mathfrak{so}(5)$ or $\mathfrak{sp}(1) + \mathfrak{sp}(p)$. The first two cases already appeared in the proof of Lemma 4.6 and the last case can be ruled out with the same arguments.

In order to complete the proof of Theorem 1.3, we need to prove the following lemma.

Lemma 4.8. *Let G be a compact Lie group acting reducibly on $\mathbb{R}^n$ $(n \geq 5)$ and coisotropically on the quadric Q_{n-2}. Then the number of irreducible submodules of $\mathbb{R}^n$ is at most two.*

Proof. We assume that $\mathbb{R}^n$ contains at least three irreducible submodules V_1, V_2, V_3, none of which is trivial. We then restrict ourselves to the coisotropic action of G on the quadric corresponding to the G-module $V = \sum_{i=1}^3 V_i$. We can apply the results obtained so far to the quadrics corresponding to the sums $V_i + V_j$, $i, j \in \{1, 2, 3\}$, showing that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$, where each $\mathfrak{g}_i \in \{\mathfrak{so}(V_i), \mathfrak{G}_2, \mathfrak{spin}(7), \mathfrak{spin}(9), \mathfrak{sp}(1) + \mathfrak{sp}(p)\}$. We consider a two plane $\pi \subset V_1$ so that $G \cdot \pi$ is a complex orbit and we consider its isotropy subalgebra $\mathfrak{g}_\pi = (\mathfrak{g}_1)_\pi + \mathfrak{g}_2 + \mathfrak{g}_3$. The slice representation of $\mathfrak{g}_\pi$ contains $Hom(\pi, V_2 + V_3)$ as a complex submodule which is coisotropic; this means that $\mathfrak{g}_\pi$ acts nontrivially on π, since otherwise $Hom(\pi, V_2 + V_3)$ would contain equivalent $\mathfrak{g}_\pi$-submodules, contradicting Proposition 1.6. The action of $\mathfrak{g}_\pi$ on $Hom(\pi, V_2 + V_3)$ is the action of $\mathfrak{h} := \mathbb{R} + \mathfrak{g}_2 + \mathfrak{g}_3$ on $(\mathbb{R}^2)^* \otimes (V_2 + V_3)$. We set $W_i := (\mathbb{R}^2)^* \otimes V_i$ and $\mathfrak{k}_i$ the regular isotropy subalgebras of $\mathbb{R} + \mathfrak{g}_i$ acting on W_i for $i = 2, 3$; it is known that the regular isotropy subalgebra of $\mathbb{R} + \mathfrak{so}(n)$ acting on $\mathbb{R}^2 \otimes \mathbb{R}^n$ is $\mathfrak{so}(n-2)$, hence $\mathfrak{k}_i \subset \mathfrak{g}_i$ for $i = 2, 3$. Now a regular isotropy subalgebra $\mathfrak{k}$ of $\mathfrak{h}$ acting on $Hom(\pi, V_2 + V_3)$ is given by $\mathfrak{k}_1 + \mathfrak{k}_2$. Since the $\mathbb{R} + \mathfrak{g}_i$-actions on W_i are coisotropic, we have that the cohomogeneity c of the $\mathfrak{h}$-action on $Hom(\pi, V_2 + V_3)$ is given by

$$\begin{aligned} c = {} & \dim(W_1 + W_2) - \dim(\mathfrak{g}_1 + \mathfrak{g}_2) - 1 + \dim(\mathfrak{k}_1 + \mathfrak{k}_2) \\ = {} & \dim W_1 - \dim(\mathbb{R} + \mathfrak{g}_1) + \dim \mathfrak{k}_1 \\ & + \dim W_2 - \dim(\mathbb{R} + \mathfrak{g}_2) + \dim \mathfrak{k}_2 + 1 \\ = {} & \operatorname{rk}(\mathbb{R} + \mathfrak{g}_1) - \operatorname{rk}(\mathfrak{k}_1) + \operatorname{rk}(\mathbb{R} + \mathfrak{g}_2) - \operatorname{rk}(\mathfrak{k}_2) + 1 \\ = {} & \operatorname{rk}(\mathbb{R} + \mathfrak{g}_1 + \mathfrak{g}_2) - \operatorname{rk}(\mathfrak{k}_1 + \mathfrak{k}_2) + 2. \end{aligned}$$

This means that the G_π-action on $Hom(\pi, V_2 + V_3)$ is not coisotropic. Hence the G-action on Q_{n-2} is not coisotropic. □

References

[Ab] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, Preprint, DG/0004122.

[AA] A. Alekseevsky and D. Alekseevsky, *Asystatic G-manifolds*, Proceedings of the Workshop on Differential Geometry and Topology, Eds. R. Caddeo, F. Tricerri, World Scientific, Singapore, 1993, pp. 1–22. MR **96f:**57035

[BR] C. Benson and G. Ratcliff, *A classification of multiplicity free actions*, J. Algebra **181** (1996), 152–186. MR **97c:**14046

[Bl] A. Blanchard, *Les variété analytique complexes*, Ann. Ec. Norm. **73** (1956), 157–202. MR **19:**316e

[BH] R. Blumenthal and J. Hebda, *De Rham decomposition theorems for foliated manifolds*, Ann. Inst. Fourier Grenoble **33** (1983), 183–198. MR **84j:**53042

[Bo] A. Borel, *Le plan projectif des octaves et les sphères comme espaces homogènes*, C. R. Acad. Sci. Paris **230** (1950), 1378–1380. MR **11:**640c

[BS] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029. *Correction* in Amer. J. Math. **83** (1961), 207–208. Also in the Collected Papers of Raoul Bott, Volume 1, 327–393 and 503–505, Birkhäuser, Boston, 1994. MR **21:**4430; MR **30:**589; MR **95i:**01027

[Br1] M. Brion, *Quelques propriétés des espaces homogènes sphériques*, Manuscripta Math. **55** (1986), 191–198. MR **87g:**14054

[Br2] M. Brion, *Classification des espaces homogènes sphérique*, Compositio Math. **63** (1987), 189–208. MR **89d:**32065

[BtD] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, Berlin-New York, 1985. MR **86i:**22023

[Co] L. Conlon, *Variational completeness and K-transversal domains*, J. Differential Geom. **5** (1971), 135–147. MR **45:**4320

[Da] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. A.M.S. **288** (1985), 125–137. MR **86k:**22019

[De] Th. Delzant, *Hamiltoniens périodiques et images convexes de l'applicationn moment*, Bull. Soc. math. France **116** (1988), 315–339. MR **90b:**58069

[Du] J.J. Duistermaat, *Convexity and tigthness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution*, Trans. A.M.S. **275** (1983), 417–429. MR **84c:**53035

[EH] J.-H. Eschenburg and E. Heintze, *On the classification of polar representations*, Math. Z. **232** (1999), 391-398. MR **2001g:**53099

[Fr] T. Frankel, *Fixed points on Kähler manifolds*, Ann. of Math. **70** (1959), 1–8. MR **24:**A1730

[GT] C. Gorodski and G. Thorbergsson, *Representations of compact Lie groups and the osculating spaces of their orbits*, Preprint.

[GS1] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping, II*, Invent. Math. **77** (1984), 533-546. MR **86b:**58042a

[GS2] V. Guillemin and S. Sternberg, *Multiplicity-free spaces*, J. Differential Geometry **19** (1984), 31–56. MR **85h:**58071

[GS3] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, 1984. MR **86f:**58054

[HPTT] E. Heintze, R. Palais, C. L. Terng and G. Thorbergsson, *Hyperpolar actions and k-flat homogeneous spaces*, J. reine angew. Math. **454** (1994), 163–179. MR **96b:**53062

[HW] A.T. Huckleberry and T. Wurzbacher, *Multiplicity-free complex manifolds*, Math. Annalen **286** (1990), 261–280. MR **91c:**32027

[Ka] V.G. Kac, *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), 190-213. MR **81i:**17005

[Ki] B. Kimelfeld, *Homogeneous domains on flag manifolds*, J. Math. Anal. Appl. **121** (1987), 506-588. MR **88h:**32027

[Kir] F. C. Kirwan, *Convexity properties of the momentum mapping, III*, Invent. Math. **77** (1984), 547–552. MR **86b:**58042b

[KN] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I, Interscience Publishers, J. Wiley & Sons, 1963. MR **27:**2945

[Ko] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, PhD thesis, Augsburg (1998).

[Kr] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Vieweg, Braunschweig, Wiesbaden, 1984. MR **86j:**14006

[Kra] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compos. Math. **38** (1979), 129–153. MR **80f:**22011

[Le] A. Leahy, *A classification of multiplicity free representations*, J. Lie Theory **8** (1998), 367–391. MR **2000g:**22024

[LE] S. Lie and F. Engel, *Theorie der Transformationsgruppen*, vol. I, Teubner, Leipzig, 1888.

[On] A. L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosius, Barth, Leipzig, Berlin, New York, 1994. MR **95e:**57058

[PT1] R.S. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. A.M.S **300** (1987), 771–789. MR **88f:**57069

[PT2] R.S. Palais and C.-L. Terng, *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics 1353, Springer-Verlag, Berlin-New York, 1988. MR **90c:**53143

[PTh] F. Podestà and G. Thorbergsson, *Polar actions on rank one symmetric spaces*, J. Differential Geom. **53** (1999), 131–175. CMP 2000:16

[Sh] A.N. Shchetinin, *On a class of compact homogeneous spaces I*, Ann. Global Anal. Geom. **6** (1988), 119–140. MR **90d:**57049

[Ya] O. Yasukura, *A classification of orthogonal transformation groups of low cohomogeneity*, Tsukuba J. Math. **10** (1986), 299–326. MR **88b:**57036

[Wo] J.A. Wolf, *Spaces of constant curvature*, (Fourth edition), Publish or Perish, Berkeley, 1977. (Fifth ed., 1984. MR **88k:**53002)

Dipartimento di Matematica e Applicazioni per l'Architettura, Università di Firenze, Piazza Ghiberti 27, I-50142 Firenze, Italy

E-mail address: `podesta@math.unifi.it`

Mathematisches Institut, Universität zu Köln, Weyertal 86-90, D-50931 Köln, Germany

E-mail address: `gthorbergsson@mi.uni-koeln.de`